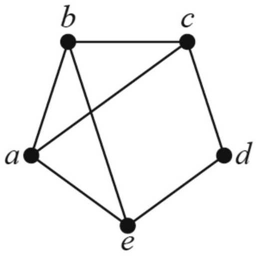
PROBLEM SET 09 - Euler walks and Hamilton paths.

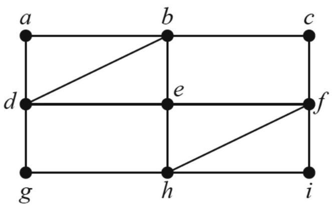
1 Euler walks and tours

Exercise 1.1. Determine whether the picture shown below can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.

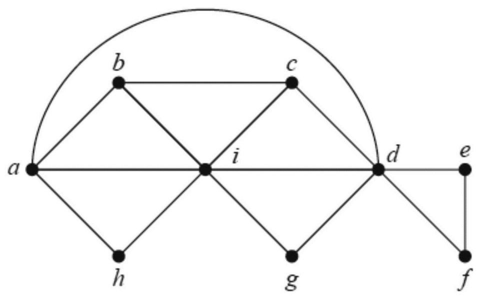


Exercise 1.2. Determine whether given graphs have an Euler tour. Construct such a tour when one exists. If no Euler tour exists, determine whether the graphs have an Euler walk and construct such a walk if one exists.

a)



b)



Exercise 1.3. For which values of n and m do these graphs have an Euler tour/walk?
a) K_n b) C_n c) W_n d) Q_n e) $K_{n,m}$

Exercise 1.4. Do all simple graphs with the given degree sequences have an Euler tour?
a) (2,2,2,2,2,2,2,2)
b) (2,2,2)
c) (2,2,2,2,2)
d) (4,4,4,4,4)
e) (6,2,2,2,2,2,2)
f) (4,4,4,4,4,4,4,4,4,4)
g) (6,4,4,4,4,4,4,4,4,4)

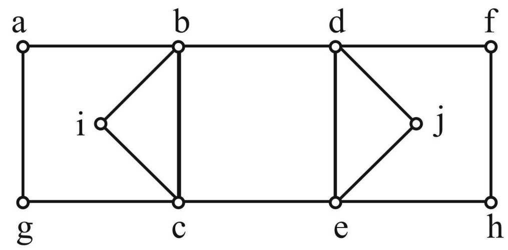
2 Fleury's algorithm

Fleury's algorithm, published in 1883, constructs Euler torus by first choosing an arbitrary vertex of a connected multi-graph, and then forming a tour by choosing edges successively. Once an edge is chosen, it is removed. Edges are chosen successively so that each edge begins where the last edge ends, and so that **this edge is not a cut edge** unless there is no alternative.

FLEURY'S ALGORITHM
INPUT: connected graph $G = (V, E)$ with all even degrees
OUTPUT: Euler tour W in G

1. Choose $v_0 \in V$ (for example the smallest in the alphabetical order)
2. $W := v_0$;
3. WHILE E is not empty
   i. $v :=$ the last vertex in W;
   ii. IF there is an edge $e = vu$ incident to v in $G = (V, E)$, which is **not a cut edge in** G choose it (for example the smallest in the alphabetical order)
   ELSE choose any edge $e = vu$ incident to v in G;
   iii. $W := Weu$;
   iv. $E := E \setminus \{e\}$;
4. RETURN W.

Exercise 2.1. Use Fleury's algorithm to find an Euler tour in the graph shown on the picture below. Consider vertices in the alphabetical order.



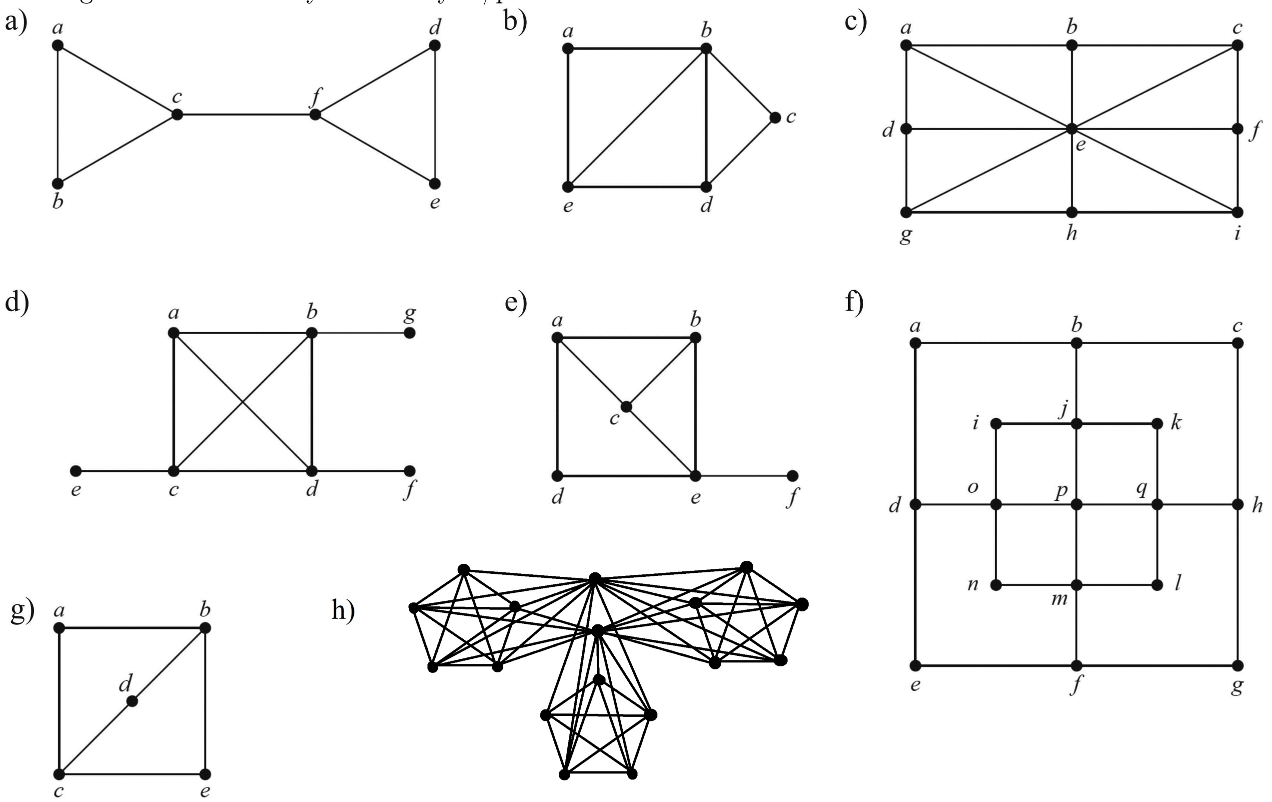
Exercise 2.2. Use Fleury's algorithm to find an Euler tour in the graphs from Exercise 1.2 which have an Euler tour.

3 Hamilton cycles

Theorem 3.1 (Dirac's Theorem)**.** *Let G be an n-vertex simple graph with $n \geqslant 3$. If $\delta(G) \geqslant n/2$, then G is Hamiltonian.*

Theorem 3.2 (Ore's Theorem)**.** *Let G be an n-vertex simple graph with $n \geqslant 3$. If for every pair of non-adjacent vertices u, v, we have $d_G(u) + d_G(v) \geqslant n$, then G is Hamiltonian.*

Exercise 3.1. Determine whether the given graph has a Hamilton cycle/path. If it does, find such a cycle. If it does not, give an argument to show why no such cycle/path exists.



Exercise 3.2. For which values of n and m do these graphs have a Hamilton cycle/path?
a) K_n b) C_n c) W_n d) Q_n e) $K_{n,m}$

Exercise 3.3. For each of the graphs from Exercise 3.2, determine
(i) whether Dirac's theorem can be used to show that the graph has a Hamilton cycle,
(ii) whether Ore's theorem can be used to show that the graph has a Hamilton cycle.

Exercise 3.4. Can you find a simple graph with n vertices, $n \geqslant 3$, that does not have a Hamilton cycle, yet the degree of every vertex in that graph is at least $(n-1)/2$?

Exercise 3.5. (*) Prove that if $G = (V, E)$ has a Hamilton cycle, then for any nonempty set $S \subseteq V$ the graph $G - S$ has at most $|S|$ connected components.

Exercise 3.6. (*) Show that the Petersen graph does not have a Hamilton cycle, but that the subgraph obtained by deleting any vertex v, and all edges incident with v, does have a Hamilton cycle.