PROBLEM SET 08 - Connectivity.

Use the following notation for:

$\omega(G)$ - the number of connected componentes of G
$\kappa(G)$ - vertex connectivity of G (the size of a minimum vertex cut of G)
$\lambda(G)$ - edge connectivity of G (the size of a minimum edge cut of G)

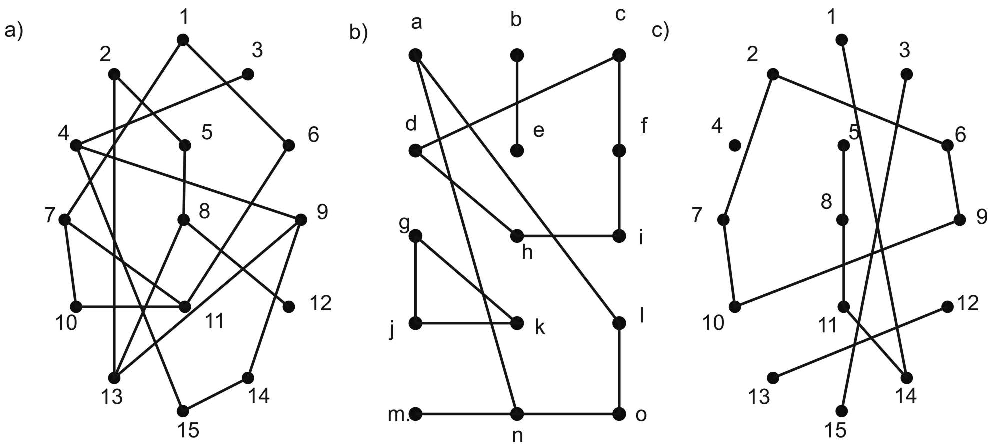
Theorem 0.1. *If $A = A(G)$ is the adjacency matrix of the graph G, then the entries of the matrix A^k, $k \in \mathbb{N}$, are equal to the number of walks of length k in G between the corresponding vertices.*

Theorem 0.2. *For any graph G*

$$\kappa(G) \leqslant \lambda(G) \leqslant \delta(G).$$

1 Connectivity/paths/cycles

Exercise 1.1. Find all connected components of the graphs given below:



Exercise 1.2. Show that every connected graph with n vertices has at least $n - 1$ edges.

Exercise 1.3. Find the number of connected components of the graphs G_1 and G_2 given by the following adjacency matrices:

$$\text{a)} \quad A(G_1) = \begin{bmatrix} 0 & A_{n\times m} \\ A_{m\times n} & 0 \end{bmatrix}, \qquad \text{b)} \quad A(G_2) = \begin{bmatrix} B_n & 0 \\ 0 & B_m \end{bmatrix}.$$

where 0 represents a matrix consisting of 0's only, $A_{k\times l}$ is a matrix with k rows and l columns consisting of 1's only, and B_k has k rows and k columns with 0's on the main diagonal and 1's beside that.

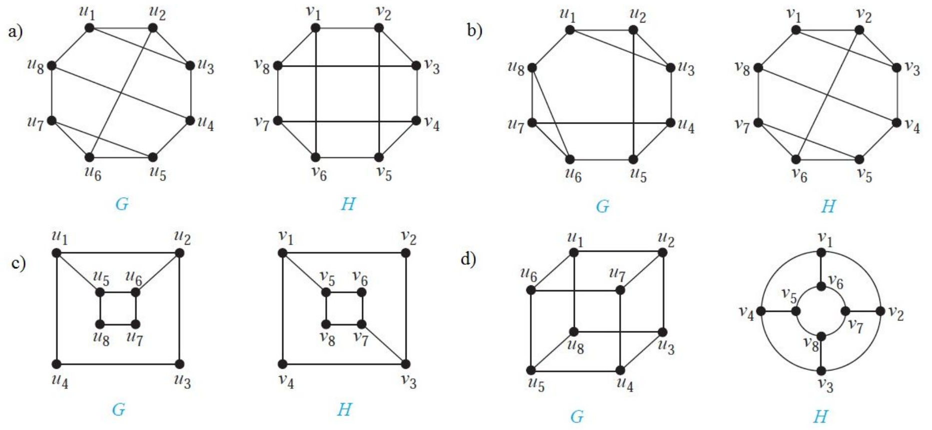
Exercise 1.4. Find the number of walks of length n between
i) two different vertices in K_4,
ii) two different nonadjacent vertices in $K_{3,3}$,
iii) two adjacent vertices in $K_{3,3}$.
if n is
a) 2, b) 3, c) 4, d) 5.
Do it directly and using the adjacency matrix.

Exercise 1.5. Generalise the results obtained in the previous exercise, i.e. find the number of walks of length k ($k \geqslant 2$) between
i) (*) two different vertices in K_n,
ii) two different nonadjacent vertices in $K_{n,n}$,
iii) two adjacent vertices in $K_{n,n}$.

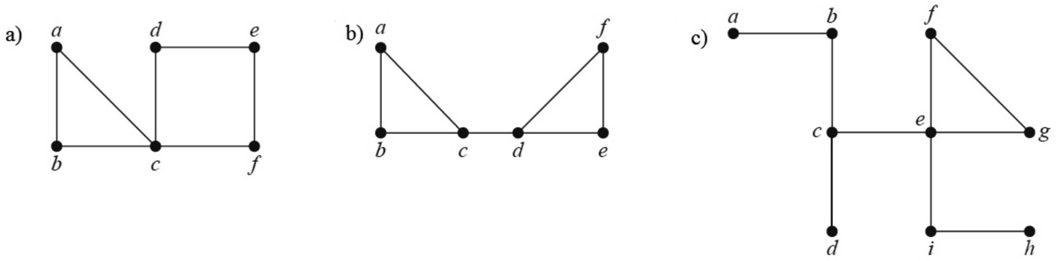
Exercise 1.6. Let A be the adjacency matrix of a graph G.
a) Using entries of A determine degrees of vertices and the number of edges in G.
b) Using entries of A^2 determine degrees of vertices and the number of edges in G.
c) Using entries of A^3 determine the number of C_3 (triangles) in G.
d) (*) Using entries of A^2 and A^4 determine the number of copies of C_4 in G.

Exercise 1.7. How many nonisomorphic connected simple graphs with n vertices are there if n is equal to
a) 2, b) 3, c) 4, d) 5.

Exercise 1.8. For each pair of graphs show that these graphs are not isomorphic or find an isomorphism between them.



Exercise 1.9. Find all cut vertices and cut edges of the given graphs



Exercise 1.10. Find κ and λ for the following graphs:
a) C_n, $n \geqslant 4$,   b) W_n, $n \geqslant 4$,   c) Q_3.

Exercise 1.11. Find $\kappa(K_{m,n})$ and $\lambda(K_{m,n})$, where m and n are positive integers.

Exercise 1.12. (*) Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

Exercise 1.13. (*) Show that a vertex c in a connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

Exercise 1.14. (*) Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.

Exercise 1.15. (*) Show that an edge in a simple graph is a cut edge if and only if it doesn't belong to any cycle in that graph.

Exercise 1.16. (*) Show that a graph $G = (V, E)$ is connected if and only if for all $V' \subseteq V$ such that $V \neq \emptyset$ and $V' \neq V$ there is an edge with one end in V' and the other in $V \setminus V'$.