# PROBLEM SET 07 - basic graph theory notions

## Use the following notation for:

$G = G(V, E)$ - a simple graph
$v_G/e_G$ - the number of vertices/edges of a graph $G$
$d_G(v)$ - the degree of a vertex $v$
$\Delta(G)/\delta(G)$ - maximal/minimal vertex degree
$G^c$ - the complement of a graph $G$
($a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and $a_{ij} = 0$ otherwise) - adjacency matrix
($a_{ij} = 1$ if $v_i$ and $e_j$ are incident and $a_{ij} = 0$ otherwise) - incidence matrix
$K_n$ - a complete graph on $n$ vertices
$C_n$ - a cycle on $n$ vertices
$P_n$ - a path on $n$ vertices
$Q_n$ - an $n$-cube (vertices – binary sequences of length $n$)
$W_n$ - a wheel graph (a vertex attached to all vertices of a cycle $C_n$)
$K_{n,m}$ - a complete bipartie graph with bipartition $(V_1, V_2)$, where $|V_1| = n, |V_2| = m$
$H \subseteq G$ - a subgraph of a graph $G$
$G[V_1]$ - a subgraph induced by a vertex set $V_1 \subseteq V$
$G[E_1]$ - a subgraph induced by an edge set $E_1 \subset E$
$G \setminus V_1$ $(V_1 \subseteq V)$ - a subgraph obtained be deleting vertices from $V_1$ together with incident edges
$G \setminus E_1$ $(E_1 \subseteq E)$ - a subgraph obtained by deleting edges from $E_1$

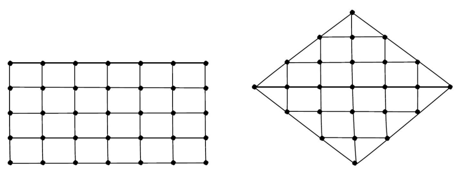
# 1 Problems

**Exercise 1.1.** For each of the graphs listed below:

a) $K_4$ b) $Q_4$ c) $K_n$ d) $Q_n$,

determine the following parameters:
- the number of vertices and edges;
- vertex degrees;
- how many edges does the complement graph have;
- the adjacency matrix.

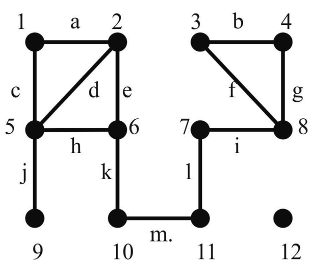
**Exercise 1.2.** Determine whether the graphs listed below are bipartite:



**Exercise 1.3.** Without drawing a picture of the graph given by the following adjacency matrix, determine the number of its vertices, its vetrex degrees and the number of edges.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

**Exercise 1.4.** Let $G$ be the following graph




and denote

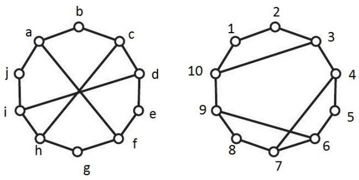
$$V_0 = \{1, 2, 5, 6, 7\} \quad \text{and} \quad E_0 = \{b, f, g, i\}.$$

(a) Draw the following graphs: $G[V_0]$, $G[V \setminus V_0]$, $G - V_0$, $G[E_0]$, $G[E \setminus E_0]$, $G - E_0$.

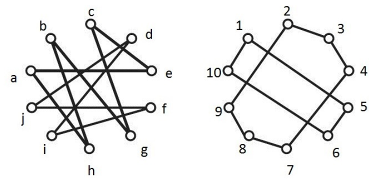
(b) Is it true that $G$ contains $K_3$, $K_4$, $C_4$, $P_4$ as subgraphs? If yes, which of those graphs are induced subgraphs?

**Exercise 1.5.** Determine whether the enclosed below pairs of graphs are isomorphic.

a)




b)




**Exercise 1.6.**

(a) Determine the number of **labeled** simple graphs on 5 vertices and with 5 edges.

(b) Draw all nonisomorphic simple graphs on 5 vertices and with 5 edges and for each drawn graph find the number of its isomorphic copies on the vertex set $\{a, b, c, d, e\}$. Check if the total number of graphs, summing over all nonisomorphic graphs, is equal to your answer given in (a).

**Exercise 1.7.** Graphs $G_1$ and $G_2$ have respective adjacency matrices:

$$A(G_1) = \begin{bmatrix} 0 & A_{n\times m} \\ A_{m\times n} & 0 \end{bmatrix} \quad \text{and} \quad A(G_2) = \begin{bmatrix} B_n & 0 \\ 0 & B_m \end{bmatrix},$$

where 0 denotes a matrix consisting fo 0's only, $A_{k\times l}$ is a matrix with $k$ rows and $l$ columns consisting only of 1's. $B_k$ is a matrix with $k$ rows and $k$ columns with 0's on the main diagonal and 1's on all other positions. Draw $G_1$ and $G_2$ for $n = 4$ and $m = 5$. For any $n$ and $m$:

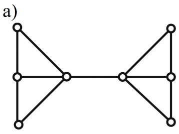
(a) determine the number of vertices and egdes of these graphs;

(b) list the vertex degrees;

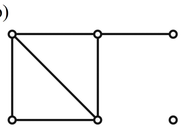
(c) check if these graphs are bipartite.

**Exercise 1.8.** For each of the graphs given below determine the number of labeled graphs isomorphic to it, i.e. determine the number of labelings of vertices of each graph $G$ by numbers $\{1, 2, \ldots, v_G\}$, such that the resulting labeled graphs are different.

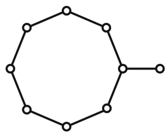
a)



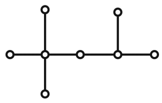
b)



c)



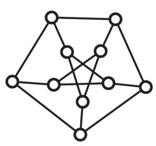
d)



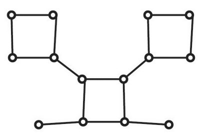
## 2 Homework

**Exercise 2.1.** For each of the graphs listed below answer the following questions:
- what is the number of vertices and edges?
- what are the vertex degrees?
- how many edges does the complement graph have?
- how does the adjacency matrix look like?
- is this graph regular?
- is this graph bipartite?

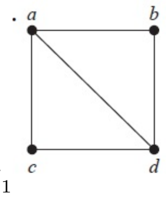
a. $C_n$ b. $P_n$ c. $K_{n,m}$ d.



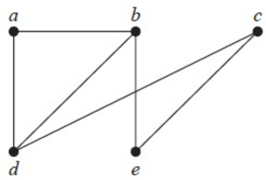
e.



**Exercise 2.2.** Find the adjacency matrix and the incidence matrix of the graphs drawn below.




$G_1$




$G_2$

**Exercise 2.3.** Without drawing the graphs, just looking at adjacency matrices given below, determine the number of their vertices and edges and their vertex degrees. Next draw corresponding graphs and check your answers.

$$a.\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad b.\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

**Exercise 2.4.** The graph $G$ has the following incidence matrix:

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $v_1$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $v_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_3$ | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v_4$ | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $v_5$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| $v_6$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $v_7$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| $v_8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

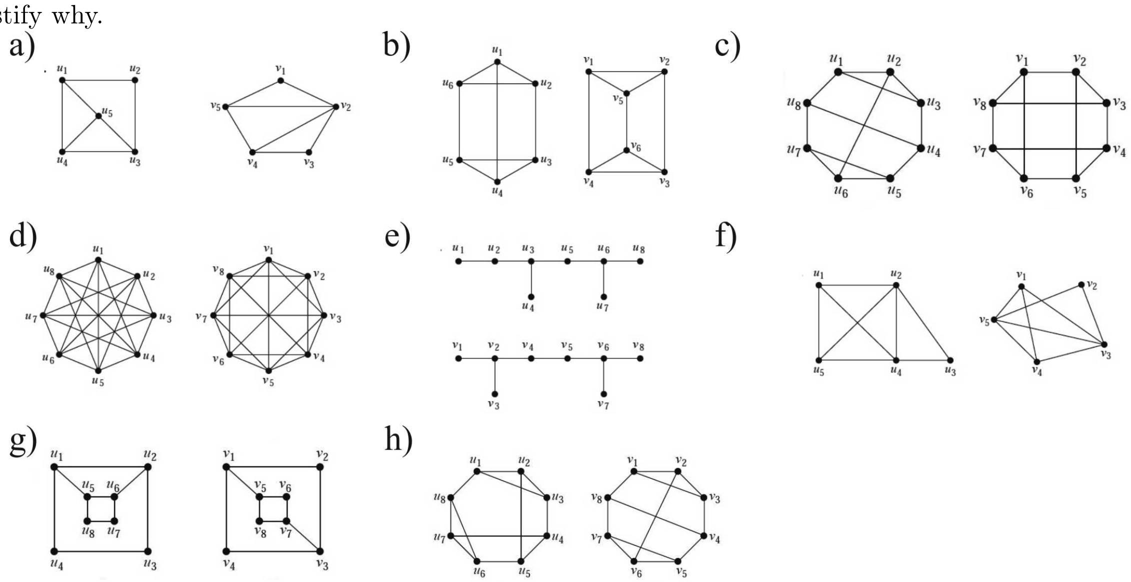
Denote

$$V_0 = \{v_4, v_6, v_7, v_8\} \quad \text{and} \quad E_0 = \{e_6, e_7, e_8, e_9\}.$$

Draw the following graphs:

$$G,\ G[V_0],\ G[V \setminus V_0],\ G - V_0,\ G[E_0],\ G[E \setminus E_0],\ G - E_0.$$

**Exercise 2.5.** For every pair of graphs given below determine whether they are isomorphic. If your answer is 'yes', show an isomorpism (provide a one-to-one mapping between the vertex sets preserving the adjacency). If your answer is 'no', justify why.




**Exercise 2.6.** A simple graph $G$ has 40 vertices. How many edges does the graph $G$ have if its complement $G^c$ has 480 edges?

**Exercise 2.7.** Draw all nonisomorphic graphs on 5 vertices which do not contain a cycle.

**Exercise 2.8.** Determine which of the graphs listed below are bipartite.