PROBLEM SET 06 - Solving recurrence relations.

Consider a linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \tag{1}$$

where c_1, c_2 are real numbers with $c_1 \neq 0$. Let r_1, r_2 be the roots of the *characteristic equation*

$$r^2 - c_1 r - c_2 = 0.$$

If $r_1 \neq r_2$, then all the solutions of (1) are of the form

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n,$$

where α_1, α_2 are constants, while for $r_1 = r_2$ the solutions of (1) are of the form

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n.$$

Similarly, for a linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3}, \tag{2}$$

with $c_1 \neq 0$ and three roots r_1, r_2, r_3 of the characteristic equation

$$r^3 - c_1 r^2 - c_2 r - c_3 = 0,$$

if all the roots are distinct then the solutions of (2) are of the form

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n,$$

if $r_1 = r_2 \neq r_3$, then the solutions of (2) are of the form

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \alpha_3 r_3^n,$$

while for $r_1 = r_2 = r_3$, the solutions of (2) are given by

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \alpha_3 n^2 r_1^n.$$

Next, consider a linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n), \tag{3}$$

and let $a_n^{(1)}$ be a solution of the linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}. \tag{4}$$

Then, the solution of (3) is of the form $a_n = a_n^{(1)} + a_n^{(2)}$. In some cases we can guess the form of $a_n^{(2)}$:

- if $f(n)$ is a polynomial od degree d and 1 is not a root of the characteristic equation of (4), then $a_n^{(2)}$ a polynomial of degree d,
- if $f(n)$ is a polynomial od degree d and 1 is a root of the characteristic equation of (4) of rank k, then $a_n^{(2)}$ a polynomial of degree $d + k$,
- if $f(n) = C\beta^n$ and β is not a root of the characteristic equation of (4), then $a_n^{(2)} = A\beta^n$,
- if $f(n) = C\beta^n$ and β is a root of the characteristic equation of (4) of rank k, then $a_n^{(2)} = An^k\beta^n$.

1 Linear homogeneous recurrence relations

Exercise 1.1. Find the solutions of the following recurrence relations.

a. $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geqslant 2$, $a_0 = 1$, $a_1 = 0$

b. $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geqslant 2$, $a_0 = 6$, $a_1 = 8$

c. $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geqslant 2$, $a_0 = 0$, $a_1 = 1$

d. $a_n = 4a_{n-2}$ for $n \geqslant 2$, $a_0 = 0$, $a_1 = 4$

e. $a_n = a_{n-1} + 6a_{n-2}$ for $n \geqslant 2$, $a_0 = 3$, $a_1 = 6$

f. $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geqslant 2$, $a_0 = 3$, $a_1 = -3$

Exercise 1.2. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

Exercise 1.3. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

Exercise 1.4. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

2 Linear nonhomogeneous recurrence relations

Exercise 2.1. Find the solutions of the following recurrence relations.

a. $a_n = a_{n-1} + 7n, a_0 = 0;$

b. $a_n = -6a_{n-1} - 9a_{n-2} + 3, a_0 = 0, a_1 = 1;$

c. $a_n = 4a_{n-1} - 4a_{n-2} + 2^n, a_0 = 2, a_1 = 2;$

d. $a_{n+1} = 2a_n - 1, a_0 = 3;$

e. $a_n = a_{n-1} + n^3, a_0 = 0.$

3 Solving complex recurrence relations

Exercise 3.1. Find the solutions of the following recurrence relations.

a. $a_n^2 = 2a_{n-1}^2 + 1, n \geqslant 1, a_0 = 2$ assuming that $a_n \geqslant 0$, for all n;

b. $a_n^2 = 2a_{n-1}, n \geqslant 1, a_0 = 4$, assuming that $a_n > 0$, for all n;

c. $a_n = \frac{1-n}{n} a_{n-1} + \frac{1}{n} 2^n, n \geqslant 1, a_0 = 3456;$

d. $a_n = na_{n-1} + n!, n \geqslant 1, a_0 = 2.$

4 Hints and solutions

Ex. 1.1 a. $a_n = 3 \cdot 2^n - 2 \cdot 3^n$
b. $a_n = 2^{n+1}(3-n)$
c. $a_n = n(-2)^{n-1}$
d. $a_n = 2^n - (-2)^n$
e. $a_n = \frac{12}{5} \cdot 3^n + \frac{3}{5} \cdot (-2)^n$
f. $a_n = (3-2n)(-3)^n$

Ex. 1.2 $a_n = -2^n - 2 \cdot (-1)^n + 6$

Ex. 1.3 $a_n = (-3)^n + 5 \cdot 2^n + 3$

Ex. 1.4 $a_n = 3 \cdot (-2)^n - 3^n + 5$

Ex. 2.1 a. $a_n = \frac{7}{2}(n^2 + n)$
b. $a_n = (-3)^n \left(-\frac{1}{12}n - \frac{3}{16}\right) + \frac{3}{16}$
c. $a_n = (2-n)2^n$
d. $a_n = 2^{n+1} + 1$
e. $a_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

Ex. 3.1 a. $a_n = \sqrt{5 \cdot 2^n - 1}$
b. $a_n = 2 \cdot 2^{\left(\frac{1}{2}\right)^n}$
c. $a_{2n-1} = \frac{2(2^{2n-1}+1)}{3(2n-1)}$, $a_{2n} = \frac{2^{2n}-1}{3n}$, $n \geqslant 1$
d. $a_n = n \cdot n!$