PROBLEM SET 10 - trees, forests, BFS, DFS

Use the following notation for:

$\tau(G)$ - the number of spanning trees of graph G
$G \cdot e$ - the graph obtained from G by contracting the edge e

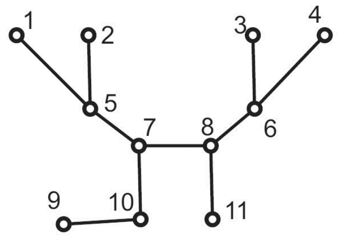
Theorem 0.1. *Let $G = (V, E)$ be a graph, and $e \in E$ be an edge which is not a loop, then*

$$\tau(G) = \tau(G - e) + \tau(G \cdot e).$$

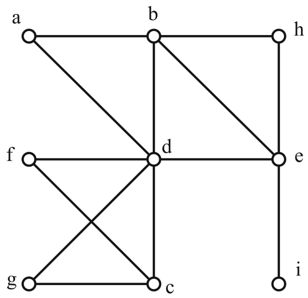
Theorem 0.2 (Cayley's formula)**.** *Let $G = (V, E)$ be the complete graph on vertex set $V = \{1, 2, \dots, n\}$. Then*

$$\tau(G) = n^{n-2}.$$

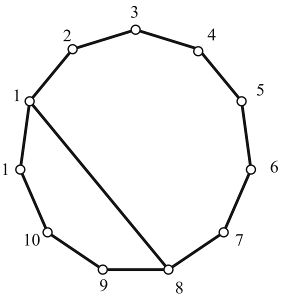
1 Trees and forests



Picture 1



Picture 2



Picture 3

Exercise 1.1. A tree T has two vertices of degree 4, one vertex of degree 3, two vertices of degree 2 and n vertices of degree 1. Find n.

Exercise 1.2. Find the Prüfer code of a tree presented in Picture 1.

Exercise 1.3. Draw (if it's possible) a tree with the following Prüfer code:

(a) (1,1,2,5,2,2,6,8)

(b) (8,1,1,1,1)

Exercise 1.4. How many trees with vertex set $\{1, 2, \dots, n\}$, $n \geqslant 3$, are there, in which

(a) vertex 1 has degree 2?

(b) with maximum degree $n - 1$?

(c) with maximum degree $n - 2$?

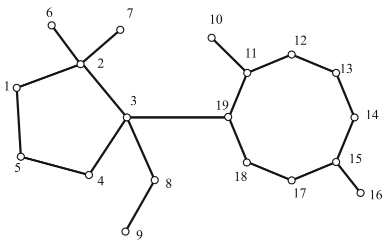
Solve the problem using Prüfer codes. Try to solve it also directly.

Exercise 1.5. Run the BFS algorithm and the DFS algorithm on a graph presented in Picture 2, starting with vertex a and in each step choosing vertices in the alphabetical order. For the BFS algorithm write down all the states of the queue and the corresponding tree, and for the DFS algorithm wirte down all the states of the stack and the corresponding tree as well.

Exercise 1.6. How many distinct spanning trees does a graph in Picture 3 have?

Exercise 1.7. How many spanning trees does $K_{2,3}$ have?

Exercise 1.8. How many spanning trees does the graph in the picture below have? What is the smallest and the biggest number of spanning trees a graph with n ($n \geqslant 5$) vertices and two edge disjoint cycles can have?



Exercise 1.9. A graph G has 225 spanning trees. After deleting an edge e, the graph $G - e$ has only 25 spanning trees. Let G^* be a graph obtained from G by subdividing e with a single vertex. How many spanning trees does G^* have?

Exercise 1.10. Describe a simple algorithm (based on BFS or DFS) which determines, whether given edge of a graph G (defined by the adjacency matrix) is a cut edge. Check if the edge v_1v_8 in the graph defined by the following adjacency matrix, is a cut edge.

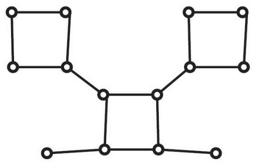
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Exercise 1.11. Is every tree a bipartite graph?

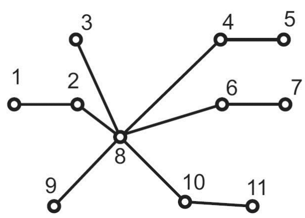
2 Homework

Exercise 2.1. How many edges one should delete from a graph with n vertices and m edges in order to obtain a spanning tree?

Exercise 2.2. How many spanning trees does K_6 and C_8 have? How about a graph in the picture below?



Exercise 2.3. Find the Prüfer code of the tree given in the picture below.



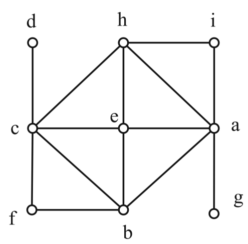
Exercise 2.4. How many trees are there with vertex set $\{1, 2, \ldots, 2n\}$ ($n \geqslant 3$) and with two vertices of degree n? Solve the problem directly and using Prüfer codes.

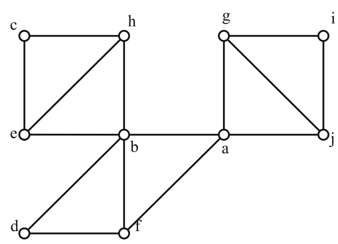
Exercise 2.5. How many different trees are there with vertex set $\{1, \ldots, 12\}$ and with the degree sequence given below (hint: use Prüfer codes)

(a) (4,4,4,2,1,1,1,1,1,1,1,1)?

(b) (5,3,2,2,2,2,1,1,1,1,1,1)?

Exercise 2.6. For the graphs given in the below pictures use the BFS algorithm and the DFS algorithm starting with vertex a and chosing vertices in the alphabetical order. For the BFS algorithm write down all the states of the queue and the corresponding tree, and for the DFS algorithm wirte down all the states of the stack and the corresponding tree as well.



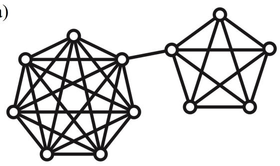


Exercise 2.7. Prove inductively, that for any tree with $n \geqslant 2$ vertices there exist at least two vertices of degree 1.

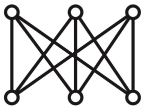
Exercise 2.8. Draw all nonisomorphic trees on 7 vertices and with maximum degree equal to 3.

Exercise 2.9. Using right theorems, find the number of spanning trees in the graphs given below.

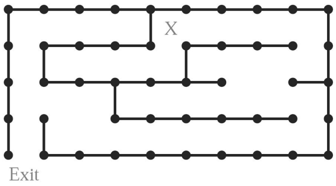
a)



b)



Exercise 2.10. Help X get out of the labirynth. Use graph representation and one of the known graph algorithms to do it.



Exercise 2.11. Both G and G^c are trees. How many vertices does G have?