

PROBLEM SET 01 - Proof methods and induction

1 Proofs

Exercise 1.1. Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a perfect square if there is an integer b such that $a = b^2$.)

Exercise 1.2. Prove by contraposition that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Exercise 1.3. Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number r and every real number s , if r and s are rational numbers, then $r + s$ is rational.")

Exercise 1.4. Prove by contradiction that if $n > 1$ and n is a non-prime natural number then it has at least one prime divisor p such that $p \leq \sqrt{n}$.

Exercise 1.5. Prove that if $m + n$ and $n + p$ are even integers, where m , n , and p are integers, then $m + p$ is even.

Exercise 1.6. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

- a proof by contraposition;
- a proof by contradiction.

Exercise 1.7. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

Exercise 1.8. Prove or disprove the statement: Every positive integer can be written as a sum of squares of three integers.

Exercise 1.9. Prove that at least one of the real numbers a_1, a_2, \dots, a_n is greater than or equal to the average of these numbers. What kind of proof did you use?

2 Induction

Basic induction

Exercise 2.1. Prove that for all $n \geq 1$ it follows that

$$7 \mid 2^{3n-1} + 3.$$

Exercise 2.2. Prove that for all $n \geq 1$ it follows that

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 2.3. Prove that for all $n \geq 3$ it follows that

$$n^{n+1} > (n+1)^n.$$

Strong induction

Exercise 2.4. Define a sequence (c_n) as follows

$$\begin{cases} c_0 = -3; \\ c_1 = 1; \\ c_n = 2c_{n-1} - c_{n-2} \quad \text{for } n \geq 2. \end{cases}$$

Prove that $c_n = 4n - 3$ for $n \geq 0$.

Exercise 2.5. Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

Exercise 2.6. Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Exercise 2.7. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

Exercise 2.8. Consider the following variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $n = 4j + 2$, or $n = 4j + 3$ for some nonnegative integer j and the second player wins in the remaining case, i.e. when $n = 4j + 1$ for some nonnegative integer j .